Example-generating tasks in a computer-aided assessment system

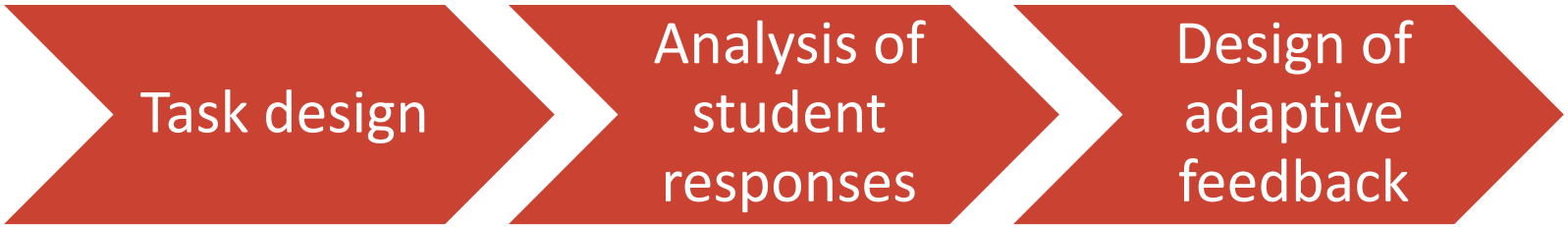
The 1st Northern e-Assessment Meeting
Trondheim, 31 May–2 June, 2023

Mats Brunström, Maria Fahlgren, Mirela Vinerean, and Yosief Wondmagegne
Karlstad University, Sweden

Introduction

- An ongoing research project with the aim to enhance first-year engineering students' engagement in and conceptual understanding of mathematics
- A combination of a *dynamic mathematics software environment* and a *computer-aided assessment system*

In focus


Task design
Analysis of student responses
Design of adaptive feedback

Example-generating tasks

Compare the following tasks:

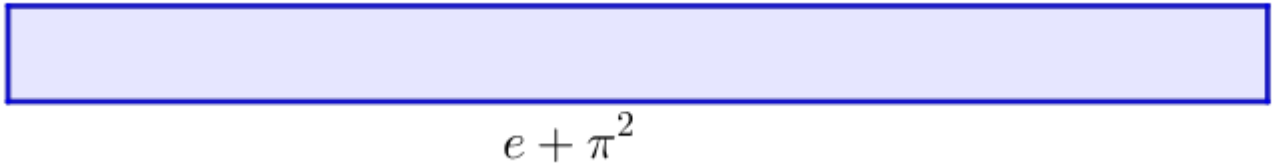
a) *Determine the area of an rectangle with a length of the 4 cm and a width of 3 cm.*

b) *Give two different examples of rectangles with the area 12 cm².*

Give two significantly different of rectangles with the area 12 cm².

Give an example that you think that no one else in the class has provided.

Give an example that you think that no one else in the world has provided.



Example spaces

- In mathematics education, examples play a key role.
- Watson and Mason (2005) advocate asking students to construct their own examples as a powerful pedagogical tool in the learning of mathematics
- They use the construct of *example spaces* when referring to collections of examples that fulfil certain conditions
- The richness of students' example space can serve as an indicator of their mathematical understanding

Theoretical constructs

- **Dimensions of possible variation (DofPV)** refer to the features of an example that are possible to vary without losing the determining characteristics.
- **The associated ranges of permissible change (RofPCh)** refer to the extent to which these dimensions can be varied while still being a valid example.

Watson, A., & Mason, J. (2005).

An exempel of an 'example-generating task'

Give examples of two different functions, f and g, both of which have

- two vertical asymptotes, $x = -6$ and $x = 3$, as well as
- a horizontal asymptote, $y = 2$.

Note:

- Group members may have received different asymptotes.
- Check in GeoGebra if your suggested functions really have the given asymptotes.

Individual response:

$f(x) =$

$g(x) =$

Data was collected from 491 students (256 students in 2020 and 235 students in 2021).

In total, 479 students (out of 491) gave a first example, among which 465 were correct. Concerning the type of function (DofPV), all students responded with rational functions.

These functions were expressed in various ways (DofPV):

Form of function formula	Example	
Single quotient	$f(x) = \frac{2x^2 + ax + b}{(x+6)(x-3)}$	**22%**
Partial fraction, reduced quotients, and a constant term (i.e. the horizontal asymptote)	$f(x) = \frac{a}{x-6} + \frac{b}{x-3} + 2$	**16%**
Reduced quotient, and a constant term (i.e. the horizontal asymptote)	$f(x) = \frac{ax + b}{(x+6)(x-3)} + 2$	**57%**

To construct the second example, most of the students (397 out of 479) changed the parameter a and/or the parameter b

Student	Asymptotes	First example	Second example
Student 1	$x = -7, x = 5$ and $y = 2$	$f(x) = \frac{2x^2 + 5x - 72}{(x+7)(x-5)}$	$g(x) = \frac{2x^2 + 8x - 78}{(x+7)(x-5)}$
Student 2	$x = -4, x = 5$ and $y = -5$	$f(x) = \frac{1}{x+4} + \frac{1}{x-5} - 5$	$g(x) = \frac{2}{x+4} + \frac{2}{x-5} - 5$
Student 3	$x = -7, x = 2$ and $y = -2$	$f(x) = \frac{1}{(x+7)(x-2)} - 2$	$g(x) = \frac{2x}{(x+7)(x-2)} - 2$
Student 4	$x = 5, x = -6$ and $y = 4$	$f(x) = \frac{1}{(x-5)(x+6)} + 4$	$g(x) = \frac{999}{(x-5)(x+6)} + 4$

Almost all students kept the degree of the numerator and denominator when constructing their second example (even if this is a DofPV). However, 20 students activated this dimension.

Asymptotes	First example	Second example
$x = -6, x = 1$ and $y = 5$	$f(x) = \frac{5(x^2+1)}{(x+6)(x-1)}$	$g(x) = \frac{5(x^2+1)^2}{\left((x+6)(x-1)\right)^2}$
$x = -4, x = 2$ and $y = 5$	$f(x) = \frac{5x^3}{(x+4)(x-2)^2}$	$g(x) = \frac{5x^4}{(x+4)^2(x-2)^2}$
$x = -7, x = 6$ and $y = 4$	$f(x) = \frac{1}{(x+7)(x-6)} + 4$	$g(x) = \frac{1}{\left((x+7)(x-6)\right)^2} + 4$
$x = -5, x = 6$ and $y = 4$	$f(x) = \frac{x}{(x+5)(x-6)} + 4$	$f(x) = \frac{(x+6)(x-5)}{(x^2-25)(x^2-36)} + 4$

How to extend students' example space?

Most of the students managed to provide two correct examples. However, almost all students provided the same type of formula in both their examples.

E.g. $f(x) = \frac{x}{(x+1)(x-2)} - 2$ and $g(x) = \frac{2x+1}{(x+1)(x-2)} - 2$

Since we think that it is instructive for students to realize that there are various ways of thinking that results in different types of formula, it would have been great if the CAA system could recognize the type of formula used by a student.

Possible feedback to extend students' example space

For example, if a student provides the responses above, the elaborated feedback could be something like:

Great, the answers are correct. Now, give one more example expressed as a single quotient.

Most of the students managed to provide two correct examples, in the form of rational functions.

Possible feedback to extend students' example space

Great, the answers are correct. Could you provide one more example that is not a rational function?

Given condition	First example	Second example
Horizontal asymptote $y = 2$	$f(x) = \frac{1}{\ln x} + 2$	$g(x) = \arctan(x) + \left(2 - \frac{\pi}{2}\right)$
Horizontal asymptote $y = 2$	$f(x) = 2 - 2^x$	$g(x) = 2^{-x} + 2$
Vertical asymptote $x = 2$	$f(x) = \frac{1}{e^{x-2} - 1}$	$g(x) = \tan\left(\frac{\pi x}{4}\right)$
Vertical asymptote $x = 2$	$f(x) = \frac{1}{2^x - 4}$	$g(x) = -\frac{1}{2^x - 4}$

Conclusion

One way to support students to extend their example space, might be to first ask for two different examples.

Then, depending on the features of these examples, students receive adapted feedback inviting them to activate a further DofPV in a third example.

A further example

Example-generating task adopted from Sangwin (2003)

Below are some possible properties (i) - (iv) of a polynomial $p(x)$.

(i) $p(x)$ is a polynomial of degree three, i.e. $p(x)$ is a cubic function.
(ii) $p(a) = 0$
(iii) $p(b) = 0$
(iv) $p(0) = ab$

a) Give an example of a polynomial $p(x)$ satisfying (i).
b) Give an example of a polynomial $p(x)$ satisfying (i) and (ii).
c) Give an example of a polynomial $p(x)$ satisfying (i), (ii) and (iii).
d) Give an example of a polynomial $p(x)$ satisfying all the properties (i) - (iv).
e) Give an example of a polynomial $p(x)$ satisfying (ii), (iii) and (iv), but not (i).

236 student responses (2021)
80% of the students provided the straightforward response to Prompt e,
i.e. the second-degree polynomial $p(x) = (x - a)(x - b)$.

The revised version of the example-generating task adopted from Sangwin (2003).

Below are some possible properties (i) - (iv) of a polynomial $p(x)$.

(i) $p(x)$ is a polynomial of degree three, i.e. p(x) is a cubic function.
(ii) $p(a) = 0$
(iii) $p(b) = 0$
(iv) $p(0) = -2ab$

a) Give an example of a polynomial $p(x)$ satisfying (i).
b) Give an example of a polynomial $p(x)$ satisfying (i) and (ii).
c) Give an example of a polynomial $p(x)$ satisfying (i), (ii) and (iii).
d) Give an example of a polynomial $p(x)$ satisfying all the properties (i) - (iv).
e) Give two examples of a polynomial $p(x)$ satisfying (ii), (iii) and (iv), but not (i).

205 students (2022)
59% provided polynomials of degree 2 and 4
14% provided polynomials of degree 4 and 5
12% provided 2 polynomials of degree 4

References

Sangwin, C. (2003). New opportunities for encouraging higher level mathematical learning by creative use of emerging computer aided assessment. *International Journal of Mathematical Education in Science and Technology*, *34*(6), 813–829.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Routledge.

Many thanks for your attention!

We are happy to discuss our ongoing work with you!